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Low energy effective action in $N = 2$ Yang-Mills as an integrated anomaly

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Abstract: Based on chiral ring relations and anomalies, as described by Cachazo, Douglas, Seiberg and Witten, we argue that the holomorphic effective action in N=2 Yang-Mills theory can be understood as an integrated U(1) anomaly from a purely field theory point of view. In particular, we show that the periods of the Riemann surface arising from the generalized Konishi anomaly can be given a physical interpretation without referring to special geometry. We also discuss consequences for the multi-instanton calculus in N=2 Yang-Mills theory.

Keywords: Nonperturbative Effects, Anomalies in Field and String Theories.

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Contents

1. Introduction

Many of the exact results for effective actions in quantum field theory can be understood as integrated anomalous Ward-identities which are protected from higher order corrections. Well known examples are the integration of the chiral (Weyl) anomaly for massless fermions in two dimensions coupled to a gauge (graviational) field or the Veneziano-Yankielowicz superpotential in $\mathcal{N} = 1$ Yang-Mills theory [1]. The integration constant in turn reflects the choice of regularization and dynamical scale. In the latter example this constant is fixed by a one-instanton calculation [2, 3]. The purpose of the present paper is to understand the exact results for the low energy effective action of $\mathcal{N} = 2$ Yang-Mills theory [4] as an integrated anomaly equation.

That the low energy effective action for $\mathcal{N} = 2$ Yang-Mills theory can be obtained by integrating the superconformal anomaly equation [5, 6] has been shown a while ago using a rather intricate series of arguments for gauge group $G = \mathrm{SU}(2)$ for pure gauge theory in [7] and including fundamental matter in [8]. For SU(2) without matter this can be understood as follows: The superconformal anomaly equation together with the $\mathrm{SL}(2, \mathbb{Z})$ structure of the mass formula implies an ordinary second order differential equation for the first derivative of the prepotential $\mathcal{F}(\mathcal{A}, \Lambda)$ for the massless $\mathcal{N} = 2$ vector multiplet $\mathcal{A}$. What is more is that the parameters of this equation are completely fixed by the weak coupling asymptotics. The integration of this equation then determines the prepotential

JHEP08(2005)006

$\mathcal{F}(\mathcal{A}, \Lambda)$ uniquely. For higher rank gauge groups, however, the superconformal anomaly equation of $\mathcal{N} = 2$ Yang-Mills is not sufficient simply because it only provides a single equation for a rank G number of low energy fields.

On the other hand, recent work pioneered by Vafa and collaborators [9 − 11] using results on geometric transitions in string theory has lead, among other results, to a new string theoretic derivation of the low energy effective action in $\mathcal{N} = 2$ Yang-Mills theory. The starting point is a certain class of $\mathrm{U}(N)$, $\mathcal{N} = 1$ theories obtained from $\mathcal{N} = 2$ Yang-Mills by adding a superpotential $W(\Phi)$ for the chiral multiplet that breaks the gauge symmetry $\mathrm{U}(N) \to \mathrm{U}(N_1) \times \cdots \times \mathrm{U}(N_n)$. This theory can be geometrically engineered via D-branes partially wrapped over certain cycles of a Calabi-Yau geometry. At low energies, the effective theory has a dual formulation where the branes are replaced by fluxes. Furthermore, the dual theory is described in terms of the gluino condensate S_i which, together with the massless $\mathrm{U}(1)$ vector multiplets w^i_α in the $\mathrm{U}(N_i)$, form an $\mathcal{N} = 2$ vector multiplet on the dual Calabi-Yau geometry [9, 10]. The holomorphic part of the effective action for this multiplet is that of an $\mathrm{U}(1)^n$, $\mathcal{N} = 2$ Yang-Mills theory spontaneously broken to $\mathcal{N} = 1$ with a superpotential $W_{\text{eff}}(S_i)$. Furthermore, this effective superpotential for the glueball superfield can be written as an integral of the holomorphic 3-form over a blown up S^3 in the Calabi-Yau geometry, or equivalently, a period integral over a dual cycle in a genus g-Riemann surface Σ. In this formulation, the effective couplings of the $\mathrm{U}(1)$ vector multiplets appear directly as the period matrix of this Riemann surface. Upon scaling the classical superpotential $W(\Phi)$ to zero one then recovers the $\mathcal{N} = 2$ theory at a point in the moduli space given by the minimum of $W(\Phi)$. Although the S_i vanish in the $\mathcal{N} = 2$ limit the structure of the Riemann surface Σ survives and the limit for the coupling τ_{ij} for the massless $\mathrm{U}(1)$'s is smooth.

In this paper we investigate how this structure can be seen to arise directly in the field theory formulation by integrating a suitable anomalous Ward-identity. This program has in fact been to a large extent completed in [12] where a certain generalization of the Konishi anomaly was used to establish that the aforementioned Riemann surface Σ arises directly in the field theory limit. We should note that the prepotential for this theory has been understood a while ago via embedding in string theory [13 − 15] and more recently through explicit multi-instanton computations to all orders [16, 17]. On the other hand, the observation that the effective action can be understood as an integrated anomaly may put this result in its proper place within the exact results for effective actions and also removes some of the mystery surrounding this model. As we will explain, it also implies that the multi-instanton calculus in this model is, in fact, equivalent to an anomalous Ward-Identity together with a one-instanton calculation and thus should elucidate the structure of the multi-instanton calculus.

The plan of this article is the following. In section 2 we review the chiral ring relations of [12] relevant for our program and discuss the consequences of these relations for instanton calculus in the $\mathcal{N} = 2$ theory. In particular we argue in section 2.3 that a specific one-instanton calculation in the chiral ring together with an anomalous Ward-Identity completely determines the n-instanton contributions to the various quantities relevant in the $\mathcal{N} = 2$ theory.

Then, in section 3 we recall first the constructions of the prepotential $F(S_i, g_k)$ and the effective couplings t_{ij} for the massless U(1) vector multiplets in terms of the Hessian of $F(S_i, g_k)$ w.r.t. S_i. The key difference with the effective prepotential $\mathcal{F}(\mathcal{A}, \Lambda)$ for the $\mathcal{N} = 2$ theory on the Coulomb branch is that while $\mathcal{F}(\mathcal{A}, \Lambda)$ has an infinite expansion in Λ and thus involves *a priori* an infinite number of instanton contributions, $F(S_i, g_k)$ is a homogenous function of the "glueball" fields S_i and the couplings g_k of the superpotential. In particular, the role of the dynamical scale Λ is reduced to setting the scale for the microscopic gauge coupling which can be evaluated in perturbation theory and a one-instanton calculation.

What remains to be shown is that the Hessian of $F(S_i, g_k)$ is given by the period matrix of the Riemann surface Σ. We should note that while in the string theory description via geometric transition [9, 10] this is an immediate consequence of special geometry, the proof in the field theory description is more intricate. In sections 3.2-3.4 we will show explicitly that this relation holds in the field theory description. This is the main technical result of this paper. For $\mathrm{U}(N)$ broken to $\mathrm{U}(1)^N$ and in the limit of vanishing superpotential, it allows us finally to express the low energy effective couplings for the massless U(1) vector multiplets in terms of the $\mathcal{N} = 2$, $\mathrm{U}(N)$ Casimirs u_k and in this way we derive the low energy effective action [4] for $\mathcal{N} = 2$ Yang-Mills theory. This is done in section 3.5. Hence, the low energy effective action has been shown to follow from integrating an anomalous Ward-identity.

2. Chiral ring relations

Our starting point is $\mathcal{N} = 2$ $\mathrm{U}(N)$ Yang-Mills broken to an $\mathcal{N} = 1$, $\mathrm{U}(1)^N$ theory. In $\mathcal{N} = 1$ superspace[1] this theory is described by a chiral multiplet Φ and a vector multiplet W_α. We add to the corresponding action a classical superpotential $\mathrm{Tr}\, W(\Phi)$ for the chiral multiplet with

$$W(\Phi) = \sum_{k=0}^{n} \frac{g_k}{k+1} \Phi^{k+1} \,. \tag{2.1}$$

As explained in [12] an important role in the field theory derivation of the glueball potential is played by the ring of $\mathcal{N} = 1$ gauge-invariant chiral operators. Since we will rely on these properties as well we present in this section a short summary of those that are relevant for us and discuss some consequences.

2.1 Definition

The chiral ring is made up of gauge invariant operators $\mathcal{O}$ which are chiral, ie. $[Q_{\dot\alpha}, \mathcal{O}\} = 0$. It follows immediately from this property that correlation functions of operators in the chiral ring in a supersymmetric vacuum ($Q_\alpha|0\rangle = 0$) are x-independent, ie.

$$-2i \frac{\partial}{\partial x_1^{\alpha\dot\alpha}} \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \langle [Q_{\dot\alpha}, [Q_\alpha, \mathcal{O}_1(x_1)\}\}\mathcal{O}_2(x_2)\rangle = 0 \,. \tag{2.2}$$

An important consequence of (2.2) is the factorization property $\langle \mathcal{O}_1\mathcal{O}_2\rangle = \langle \mathcal{O}_1\rangle\langle \mathcal{O}_2\rangle$. Another property of chiral operators is that in a supersymmetric vacuum the expectation

[1] We use the conventions of Wess and Bagger [18].

values of $\mathcal{O}$ and $\mathcal{O} + Q_{\dot\alpha}X^{\dot\alpha}$ are identical provided $X^{\dot\alpha}$ is some gauge invariant operator. Thus, we consider the chiral ring as being generated by the equivalence classes

$$\mathcal{O} \simeq \mathcal{O} + Q_{\dot\alpha}X^{\dot\alpha}\,. \tag{2.3}$$

One can represent (not necessarily gauge invariant) chiral operators as the lowest components of chiral superfields Ψ (since $[Q_{\dot\alpha}, \Psi\}|_{\theta=0} = [D_{\dot\alpha}, \Psi\}|_{\theta=0}$). Then, using

$$\bar\nabla^2 \nabla_\alpha \Psi = [W_\alpha, \Psi\}\,, \tag{2.4}$$

where ∇_α is the super- and gauge covariant derivative, one finds that W_α (anti) commutes with Ψ in the chiral ring. Taking this into account one sees that the $\mathcal{N}=1$ chiral ring is generated by the elements

$$\{\mathrm{Tr}(\Phi^k), \mathrm{Tr}(\Phi^k W_\alpha), \mathrm{Tr}(\Phi^k W_\alpha W^\alpha)\}\,. \tag{2.5}$$

In the rest of this paper certain relations between elements of the chiral ring will play a crucial role. In order to describe these relations for the complete set of elements simultaneously it is useful to introduce a generating function for all elements of the chiral ring. Such a generating function is given by

$$\mathcal{R}(z,\psi) \equiv \mathrm{Tr}\,\hat{\mathcal{R}}(z,\psi) = \frac{1}{2}\,\mathrm{Tr}\left(\left(\frac{1}{4\pi}W^\alpha - \psi^\alpha\right)^2 \frac{1}{z-\Phi}\right), \tag{2.6}$$

where $z \in \mathbb{C}$ and ψ_α is a Grassman-valued parameter. The various elements of the chiral ring are then given by the coefficients in the expansion of $\langle \mathcal{R}(z,\psi)\rangle$ in powers of ψ_α and $\frac{1}{z}$.

2.2 Classical and quantum relations

As explained in the introduction the basic idea we will employ is to integrate an anomalous Ward identity in order to obtain the effective action for the massless degrees of freedom in the Coulomb branch of $\mathcal{N}=2$ Yang-Mills theory. Concretely this anomaly is manifested as a quantum correction to a classical relation in the chiral ring. The relation we consider is simply a consequence of the equation of motion, $\bar\nabla^2\bar\Phi = \partial_\Phi W(\Phi)$, where $W(\Phi)$ is the classical superpotential. Since classically $\bar\nabla_{\dot\alpha}$ commutes with any chiral field we can multiply this equation by $\hat{\mathcal{R}}$ to get

$$\bar D^2\,\mathrm{Tr}\left(\hat{\mathcal{R}}(z,\psi)\bar\Phi\right) = \mathrm{Tr}\left(\hat{\mathcal{R}}(z,\psi)\partial_\Phi W(\Phi)\right). \tag{2.7}$$

Upon quantization, normal ordering effects have to be taken into account which lead to an anomaly in the above relation [19, 20]. This anomaly can be determined, for instance, using Pauli-Villars regularisation (e.g. [21]). In perturbation theory this one loop calculation is exact. It was shown in [22] that for the types of superpotentials considered here there are no non-perturbative corrections to the anomaly. The anomalous relation is then given by [12]

$$\bar D^2\,\mathrm{Tr}\left(\hat{\mathcal{R}}(z,\psi)\bar\Phi\right) = \mathrm{Tr}\left(\hat{\mathcal{R}}(z,\psi)\partial_\Phi W(\Phi)\right) + \frac{1}{32\pi^2}\sum_{kl}\left[W_\alpha, \left[W^\alpha, \frac{\partial\hat{\mathcal{R}}}{\partial\Phi_{kl}}\right]\right]_{lk}, \tag{2.8}$$

where Φ_{kl} are the entries of the matrix Φ. Note that the above equation still holds if we replace W_α by $W_\alpha - 4\pi\psi_\alpha$ since ψ_α (anti) commutes with everything. Still following [12] and using the identity

$$\sum_{kl}\left[\chi_1,\left[\chi_2,\frac{\partial}{\partial\Phi_{kl}}\frac{\chi_1\chi_2}{z-\Phi}\right]\right]_{lk} = \mathrm{Tr}\left(\frac{\chi_1\chi_2}{z-\Phi}\right)\mathrm{Tr}\left(\frac{\chi_1\chi_2}{z-\Phi}\right), \tag{2.9}$$

valid in the chiral ring for anti commuting operators χ_1 and χ_2 and taking expectation values, we have

$$\langle\mathcal{R}(z,\psi)\mathcal{R}(z,\psi)\rangle = \mathrm{Tr}\,\langle\hat{\mathcal{R}}(z,\psi)\partial_\Phi W(\Phi)\rangle\,. \tag{2.10}$$

Finally, recalling the factorization properties of the chiral ring this leads to

$$\langle\mathcal{R}(z,\psi)\rangle^2 = \partial_z W(z)\langle\mathcal{R}(z,\psi)\rangle + \frac{1}{4}f(z,\psi) \tag{2.11}$$

where $f(z,\psi) = \sum_{i=0}^{n-1} f_i(\psi)z^i$ is a polynomial of order $n-1$ in z defined by

$$f(z,\psi) = -2\left\langle \mathrm{Tr}\left(\frac{(W'(z)-W'(\Phi))(\frac{W^\alpha}{4\pi}-\psi^\alpha)^2}{z-\Phi}\right)\right\rangle. \tag{2.12}$$

The coefficients $f_i(\psi)$ are thus given by

$$f_i = -\frac{1}{2\pi}\sum_{q=i+1}^{n} g_q t_{q-i-1} \tag{2.13}$$

where we have defined

$$t_k(\psi) = \frac{1}{4\pi}\left\langle\mathrm{Tr}\left(\phi^k(W^\alpha - 4\pi\psi^\alpha)^2\right)\right\rangle. \tag{2.14}$$

Eq. (2.11) is then solved for $\langle\mathcal{R}(z,\psi)\rangle$ as

$$2\langle\mathcal{R}(z,\psi)\rangle = W'(z) - \sqrt{W'(z)^2 + f(z,\psi)}\,. \tag{2.15}$$

In the above expectation values it is understood that the high energy degrees of freedom have been integrated out while the low energy degrees of freedom play the role of background fields which are assumed to take values compatible with unbroken supersymmetry, so that the chiral ring properties are realized. It has been shown in [23] that each solution of the chiral ring relations corresponds to a supersymmetric vacuum of the gauge theory. In what follows we consider the case where the $\mathrm{U}(N)$ gauge symmetry is broken down to $\mathrm{U}(1)^N$. In analogy with the case where the breaking is to a product of $\mathrm{U}(N_i)$ groups, we take the low energy fields to be the massless $\mathrm{U}(1)_i$ vector multiplets w_i^α together with the glueball superfields[2] S_i with the condensate of the massless gluinos as their lowest components. In particular, the chiral multiplet which acquires a mass due to the classical

[2]It is generally assumed that the glueball field is the appropriate variable in the low energy theory although we are not aware of a rigorous justification of this assumption within field theory (see also [24]).

superpotential $W(\Phi)$ is integrated out. The light degrees of freedom are then conveniently combined with the help of the auxiliary variable ψ_α as

$$\begin{aligned}\mathcal{S}_i &= S_i - w_i\psi + \frac{1}{2}\psi^2 N_i \\ &= \frac{1}{2\pi i}\oint_{A_i}\langle\mathcal{R}(z,\psi)\rangle\,,\end{aligned} \tag{2.16}$$

where the contour A_i is around the cut extending from the i-th minimum of the function $W(z)$ which are assumed to be non-degenerate. The details concerning the projection onto S_i in terms of the contour integrals can be found in [12].

It follows from (2.16) that the expectation value $\langle\mathcal{R}(z,\psi)\rangle$ depends on ψ_α only implicitly through the $\mathcal{S}_i$. Furthermore, defining

$$\mathcal{Y}(z,\psi) = \sqrt{W'(z)^2 + f(z,\psi)} \tag{2.17}$$

and using eq. (2.15) gives

$$\mathcal{S}_i = -\frac{1}{4\pi i}\oint_{A_i}\mathcal{Y}\,. \tag{2.18}$$

Note also that $\mathcal{S}_i$ can be considered as a function of the variables g_k and $f_j(\psi)$ and conversely $f(z,\psi) = f(z,\mathcal{S}_j(\psi),g_k)$.

The result (2.15), obtained in [12] shows the appearance of a genus $g = N-1$ Riemann surface Σ defined by eq. (2.17) (evaluated at $\psi_\alpha = 0$) as a consequence of the anomalous relation (2.8) in the chiral ring. The period matrix $t_{ij}(S_i, g_k)$ of Σ will be identified with the low energy effective couplings $\tau_{ij}(u_k)$ of the massless fields in the Coulomb branch of the $\mathcal{N} = 2$ theory. In order to establish this result we first need to understand how $t_{ij}(S_i, g_k)$ enters in the effective action for the massless vector multiplets $w_{i\alpha}$. This is done in section 3.

2.3 Chiral ring relations and instanton calculus

As explained above the expectation values in eq. (2.11) stand for integrating out the high energy degrees of freedom in the presence of the "external" fields S_i and $w_{i\alpha}$. However, another interpretation is possible in which all degrees of freedom are integrated out. In this case the coefficients f_i defined by eq. (2.13) are functions of g_k only. If the superpotential is such that the gauge symmetry is broken to $\mathrm{U}(1)^N$, then there is no strong infrared dynamics such that the coefficients f_i can be evaluated by treating the superpotential perturbatively in the $\mathcal{N} = 2$ theory. In addition f_i, or rather $t_k = \frac{1}{4\pi}\mathrm{Tr}\,\langle\phi^k W^2\rangle$ (see eq. (2.14)) is saturated by classical and instanton contributions. This is because t_k is annihilated by $\bar{Q}_{\dot\alpha}$ implying that the path integral localizes on just these configurations. Concretely we have for the s-instanton contribution

$$t_k = \sum_{r_0,\cdots,r_N} a_{s,r_0,r_1,\cdots,r_N}\Lambda^{2Ns} g_0^{r_0}\cdots g_N^{r_N}\,, \tag{2.19}$$

with $r_l \geq 0$ and subject to the constraints $\sum_l r_l = 1$ and $2Ns - \sum_l r_l(l+1) = k$. These selection rules can be understood from symmetries [12] or alternatively by looking at the details of the s-instanton calculation: Since we are expanding about the theory with vanishing superpotential ($g_k = 0$) only non-negative powers of g_k appear in the expansion.[3] In order to get a non-vanishing contribution from an instanton with topological charge s we need to saturate the $4Ns$ fermionic zero modes: Two gluino zero modes appear in $\mathrm{Tr}\left(\phi^k\lambda^2\right)$. Further zero modes appear by expanding in terms of the Yukawa coulings $\mathrm{Tr}(\lambda\phi^\dagger\psi)$. It is then not hard to see that in order to produce $2Ns$ gluino λ and "quark" ψ zero modes we need to have exactly one contribution $g_l\,\mathrm{Tr}(\phi^{l-1}\psi^2)$ from the potential $\int d^2\theta W(\Phi)$. This explains the first selection rule given above. Furthermore, the total number of scalars contained in t_k and $g_l\,\mathrm{Tr}(\phi^{l-1}\psi^2)$ must match the $2Ns-2$ scalars coming from the expansion in the Yukawa couplings. This then leads to the second selection rule stated above which now has the interpretation of a tree level amplitude in the presence of $4Ns$ fermionic zero modes. A consequence of these selection rules is that the t_k's vanish for $0 \leq k < N-1$ and t_{N-1} is determined by a 1-instanton contribution.

For $k \geq N$ the connected part of $t_k = \frac{1}{4\pi}\langle\,\mathrm{Tr}(W^2\Phi^k)\rangle$ is still subject to the selection rules above and similarly for the connected part of $u_k = \langle\,\mathrm{Tr}\,\Phi^k\rangle$, that is

$$\langle\,\mathrm{Tr}\,\Phi^k\rangle_c = \sum_s a_s\Lambda^{2Ns}\,. \tag{2.20}$$

Here the connected part of the expectation value is the part that cannot be written as a product of expectation values with less than k fields. In particular, for $k < 2N$, u_k is fully determined by its classical expression in terms of the g_k's.

Combining the selection rules for (2.19) and (2.20) with the chiral ring relation (2.15) one then concludes that the expansions of t_k and u_k in the dynamical scale Λ are completely determined by a single 1-instanton calculation and the classical expressions for the u_k, $k \leq N$. Indeed, expanding both sides of eq. (2.11) in $1/z$ we obtain the recurrence relation

$$\sum_{p,q=0}^{\infty}\mathcal{R}_p\mathcal{R}_q z^{-p-q-2} = \sum_{q=0}^{\infty}\sum_{p=0}^{N} g_p\mathcal{R}_q z^{-p-q-1} \tag{2.21}$$

where $\langle\mathcal{R}(z,\psi)\rangle = \sum z^{-q}\mathcal{R}_q$. The recursion relation is then obtained by comparing equal powers in z (see also [23] for a related discussion).

An important application of this discussion is that, of the coefficients f_i, only f_0 is non-vanishing [12] and is then equal to $d_0 g_N^2\Lambda^{2N}$. The factor d_0 can in principle be determined by evaluating t_{N-1} in the explicit one-instanton computation. Instead we will determine d_0 below by comparison with the 1-instanton contribution to the superconformal anomaly in the $\mathcal{N} = 2$ theory. This, with the help of (2.15), fixes $\langle\mathcal{R}(z,\psi)\rangle$ and, in particular, S_i completely in terms of the g_k and the one-instanton coefficient f_0. This suggests that the chiral ring relations, or equivalently the Konishi anomaly imply a relation between instanton contributions of different topological charges (see [27–29] for other approaches to instanton recursion relations).

[3] This is in contrast to explicit instanton calculations in the Coulomb branch of $\mathcal{N} = 2$ Yang-Mills theory (see [3, 25, 26, 16, 17] and references therein) where gauge and dilatation symmetry is spontaneously broken from the outset.

3. Effective action and low energy couplings

In order to determine the couplings of the massless U(1) vector multiplets w^i_α, we will need to determine the holomorphic part of the effective action for these fields. In the $\mathcal{N}=2$ theory this action is usually expressed in terms of a prepotential $\mathcal{F}(\mathcal{A},\Lambda)$ for the $\mathcal{N}=2$ vector multiplet $\mathcal{A}$. However, as explained in the introduction this prepotential involves an infinite number of instanton contributions. The key observation [11] allowing to circumvent this problem is that the low energy effective couplings can equally be obtained from an effective superpotential $W_{\rm eff}(g_k, S_i, w_{i\alpha})$ which in contrast to $\mathcal{F}(\mathcal{A},\Lambda)$ does not receive higher instantons contributions.

3.1 Effective action

In analogy with the usual effective action in field theory the holomorphic part of the effective action is given by the sum over 1PI graphs with S_i and $w_{i\alpha}$ insertions. In fact we can say more. A "non-renormalisation theorem" given in [12] shows that in perturbation theory the only contributions, compatible with the expected symmetries of the effective action come from planar graphs. Furthermore these planar graphs have either exactly one $w_{i\alpha}$ insertion at two of the index loops and one S_i insertion at each of the remaining index loops or one S_i insertion in all but one index loop. The index loop without S_i insertion being proportional to N_i, where

$$N_i = \frac{1}{2\pi i}\oint_{A_i}\left\langle \mathrm{Tr}\,\frac{1}{z-\Phi}\right\rangle \tag{3.1}$$

counts the degeneracy of the vacuum corresponding to the $i-th$ minimum of the potential. This observation then implies that the holomorphic part of the effective action for the low energy fields can be expressed in terms of a single function $F(g_k, S_i)$. Indeed let $F(g_k, S_i)$ be the sum over all 1PI graphs with exactely one S_i insertion at each of the index loops. Then, since the effective potential is obtained by either replacing one S_i by N_i or two S_i insertions by $w_{i\alpha}$'s, we have

$$W_{\rm eff}(g_k, S_i, w_{i\alpha}) = \sum_i N_i \frac{\partial F(g_k,S_i)}{\partial S_i} + \sum_{ij}\frac{\partial^2 F(g_k,S_i)}{\partial S_i \partial S_j} w_{i\alpha} w_j^\alpha \,. \tag{3.2}$$

This latter relation can be written equivalently as

$$W_{\rm eff}(g_k, S_i, w_{i\alpha}) = \mathcal{F}(g_k, \mathcal{S}_i)\Big|_{\psi^2} . \tag{3.3}$$

where $\mathcal{F}(g_k, \mathcal{S}_i)$ is obtained from $F(g_k, S_i)$ simply by the substitution $S_i \to \mathcal{S}_i$. In order to evaluate the coupling for the massless U(1) fields $w_{i\alpha}$ we then set the S_i "on shell". There are two equivalent ways of implementing this. In the last section we have seen that on shell the f_i's are expressed in terms of g_k. On the other hand, from the point of view of $W_{\rm eff}(g_k, S_i, w_{i\alpha})$ this means that

$$\frac{\partial W_{\rm eff}(g_k, S_i, w_{i\alpha}=0)}{\partial S_i} = 0 \tag{3.4}$$

so that the coupling matrix is

$$t_{ij}(S_i, g_k) = \frac{\partial^2 F}{\partial S_i \partial S_j}\,. \tag{3.5}$$

However, to compare (3.5) with the couplings for the massless U(1) vector multiplets in SU(N) $\mathcal{N}=2$ Yang-Mills we need to change variables from the $w_{\alpha i}$ to usual Cartan-Weyl basis for SU(N). We will call these variables α_i, $i=1,\dots,N-1$. They are defined by

$$\alpha_i = S_i - \frac{1}{N}\sum_{j=1}^{N} S_j \qquad \text{for } i=1,\dots,N-1 \tag{3.6}$$

while $\alpha_+ = \frac{1}{N}\sum_{i=1}^{N} S_i$. In particular, α_+ vanishes on shell. The couplings for the α_i are then given by $\tau_{ij}(g_k, S_i) = \frac{\partial^2 F}{\partial \alpha_i \partial \alpha_j}$.

In principle $F(g_k, S_i)$ can now be computed directly order by order in perturbation theory[4] [30] (or in an associated matrix model [31]). However, for the couplings, we only need its second derivative $F(g_k, S_i)$ with respect to S_i. We will now show explicitely that the Hessian of F can be expressed in terms of the period matrix of the Riemann surface described by (2.17). This is the main technical result of the present paper.

3.2 Effective action and chiral ring

First we recall a result in [12] relating the derivative of $F(g_k, S_i)$ with respect to g_k to the expectation value of $\mathcal{R}(z,\psi)$ introduced in section 2. To start with we have from eq. (2.1)

$$\frac{\partial W_{\mathrm{eff}}(g_k, S_i, w_{i\alpha})}{\partial g_k} = \frac{1}{(k+1)}\left\langle \mathrm{Tr}\left(\Phi^{k+1}\right)\right\rangle_{\{S_i, w_{j\alpha}\}}\,. \tag{3.7}$$

On the other hand $W_{\mathrm{eff}}(g_k, S_i, w_{i\alpha})$ is expressed in terms of $F(g_k, S_i)$ via eq. (3.3). The claim is that this implies

$$\frac{\partial \mathcal{F}(g_k, \mathcal{S}_i)}{\partial g_k} = -\frac{1}{2(k+1)}\left\langle \mathrm{Tr}\left(\Phi^{k+1}\left(\frac{W_\alpha}{4\pi} - \psi_\alpha\right)\left(\frac{W^\alpha}{4\pi} - \psi^\alpha\right)\right)\right\rangle_{\{S_i, w_{j\alpha}\}}\,. \tag{3.8}$$

The identity (3.8) follows from the observation that the right hand side depends on the background fields $\{S_i, w_{j\alpha}\}$ only in the combination $\mathcal{S}_i$. On the other hand (3.3) and (3.7) imply that the ψ^2 component of both sides of the equation (3.8) agree. Finally, by construction both sides vanish at $\mathcal{S}_i = 0$. Consequently the two functions are the same.

We will now show that $\frac{\partial F(g_k, S_i)}{\partial \alpha_i}$ is expressed in terms of the (dual) B_i-periods of the Riemann surface Σ defined by eq. (2.17). In the string theory description of this model [10] this property follows directly from the special geometry of CY spaces. In field theory we are not aware of a previous derivation of this property (see also comment at the end of

[4] As already emphasized in the introduction $F(g_k, S_i)$ is a homogenous function in g_k and S_i. In particular the instanton contributions are summarized in the g_k-independent integration constant which is fixed by a one-instanton calculation. See also [24] for a discussion of the ambiguities in the field theory computation of $F(S_i, g_k)$.

section 4 in [12]). The strategy we use is to first compute $\frac{\partial^2 F(g_k,S_i)}{\partial S_i \partial g_k}$ and then to show that the result obtained can be integrated to obtain $\frac{\partial F(g_k,S_i)}{\partial S_i}$ up to an integration "constant" independent of g_k. We start with eq. (3.8) which gives for $\psi_\alpha = 0$

$$\begin{aligned}\frac{\partial F(g_k,S_i)}{\partial g_k} &= -\frac{1}{32\pi^2(k+1)}\left\langle \mathrm{Tr}\left(\Phi^{k+1}W_\alpha W^\alpha\right)\right\rangle_{S_i} \\ &= -\frac{1}{2}\frac{1}{(k+1)}\mathrm{Res}_\infty\left[z^{k+1}y(z)\right], \end{aligned} \tag{3.9}$$

where we have used eq. (2.6), (2.15) and $y(z) = \mathcal{Y}(z)|_{\psi_\alpha=0} = \sqrt{W'(z)^2 + f(z)}$. The next step is then to characterize completely $y(z)$. As it is a mereomorphic 1-form with a singular point of order N at infinity, it can be written as [32]

$$y = \sum_{k=0}^{N} \beta_k d\Omega_k + \beta_+ d\Omega_+ + \sum_{i=1}^{N-1} \gamma_i \xi_i \tag{3.10}$$

with the following definitions: ξ_i are canonically normalized holomorphic 1-forms such that $\oint_{A_i} \xi_j = \delta_{ij}$. $d\Omega_k$ behave at infinity like $d\Omega_k = z^k + O(z^{-2})$ and satisfy[5] $\oint_{A_i} d\Omega_k = 0$ for all i. We have similarly $d\Omega_+ = z^{-1} + O(z^{-2})$ and $\oint_{A_i} d\Omega_+ = 0$ for $i = 1, \ldots, N-1$.

Let us now determine the coefficients β_k, β_+ and γ_i. The coefficient γ_i is obtained by integrating around the contour A_i. It follows from (2.18), the previous definitions and from (3.6) that $\gamma_i = -4\pi i(\alpha_i + \alpha_+)$. The coefficient β_k is obtained by considering the behavior of y at infinity. More precisely, as

$$y(z) = W'(z)\sqrt{1 + \frac{f(z)}{W'^2(z)}} = W'(z)(1 + O(z^{-N-1})), \tag{3.11}$$

we have $\mathrm{Res}_\infty(z^{-k-1}y) = g_k$ for $0 \le k \le N$ which leads to $\beta_k = g_k$. Finally, we have $\beta_+ = -2N\alpha_+$ as

$$\beta_+ = \frac{1}{2\pi i}\oint_\infty y = \frac{1}{2\pi i}\sum_{i=1}^{N}\oint_{A_i} y = -2\sum_{i=1}^{N} S_i = -2N\alpha_+ . \tag{3.12}$$

Thus we have shown that

$$y = \sum_{k=0}^{N} g_k d\Omega_k - 4\pi i \sum_{i=1}^{N-1}\alpha_i \xi_i - \alpha_+\left(2N\, d\Omega_+ + 4\pi i \sum_{i=1}^{N-1}\xi_i\right). \tag{3.13}$$

We prove then in appendix A that y is homogenous of degree one in α_i, α_+ (or S_i) and g_k. In particular, we have

$$\frac{\partial y}{\partial g_k} = d\Omega_k \qquad \text{and} \qquad \frac{\partial y}{\partial \alpha_i} = -4\pi i \xi_i . \tag{3.14}$$

[5] Note that this can always be achieved by adding a suitable linear combination of holomorphic 1-forms.

After these preparations we consider $\partial^2 F/\partial g_k \partial \alpha_i$. We get then from equations (3.9) and (3.14)

$$\begin{aligned}\frac{\partial^2 F}{\partial g_k \partial \alpha_i} &= \frac{\partial^2 F}{\partial \alpha_i \partial g_k} = -\frac{1}{2}\frac{1}{k+1}\mathrm{Res}_\infty\left(z^{k+1}\frac{\partial y}{\partial \alpha_i}\right), \\ &= 2\pi i \frac{1}{k+1}\mathrm{Res}_\infty(z^{k+1}\xi_i)\,. \end{aligned} \tag{3.15}$$

In the next subsection we integrate this relation.

3.3 Identification of the B_i periods

We will now show that the expression (3.15) can be written as a derivative with respect to g_k of an integral over the cycle B_i dual to A_i. For this, we use the Riemann bilinear relations associated with one form of first kind (i.e. holomorphic), ξ_i, and the other of second kind (i.e. meromorphic with no residues), $d\Omega_k$:

$$\sum_{j=1}^{N-1}\left[\oint_{A_j} d\Omega_k \oint_{B_j}\xi_i - \oint_{A_j}\xi_i \oint_{B_j} d\Omega_k\right] = 2\pi i\left[\mathrm{Res}_P(\Omega_k\xi_i) + \mathrm{Res}_{\tilde P}(\Omega_k \xi_i)\right]. \tag{3.16}$$

Let us recall briefly that such relations are obtained by computing in two different ways one particular integral [33], $\int_{\partial\Sigma}\Omega_k\xi_i$, where the Riemann surface Σ is thought here as a polygon with some identifications (see figure 1). Then, the l.h.s. corresponds to the computation of this integral on the contour shown on figure 1 while the r.h.s. of eq. (3.16) is simply obtained by use of Cauchy's formula where P and $\tilde P$ have coordinates $z(P) = z(\tilde P) = \infty$ with $y(P) = -y(\tilde P)$.

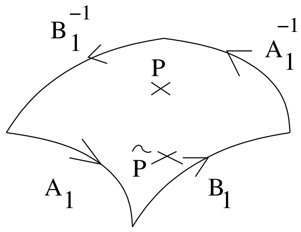

Figure 1: Riemann bilinear relation.

However, for the l.h.s. we use the definitions of ξ_i and $d\Omega_k$, which give respectively $\oint_{A_j}\xi_i = \delta_{ij}$ and $\oint_{A_j} d\Omega_k = 0$. For the r.h.s., we have $\mathrm{Res}_P(\Omega_k\xi_i) = \mathrm{Res}_{\tilde P}(\Omega_k\xi_i)$ because both $d\Omega_k = \partial y/\partial g_k$ and ξ_i change sign when going from P to $\tilde P$. Furthermore these residues are equal to $\frac{1}{k+1}\mathrm{Res}_\infty(z^{k+1}\xi_i)$ as $d\Omega_k = z^k + O(z^{-2})$ and $\xi_i = O(z^{-2})$. Using the result (3.14) we finally get

$$-\frac{1}{2}\oint_{B_i}\frac{\partial y}{\partial g_k} = 2\pi i \frac{1}{k+1}\mathrm{Res}_\infty(z^{k+1}\xi_i)\,. \tag{3.17}$$

This enables us to write eq. (3.15) as $\frac{\partial^2 F}{\partial g_k \partial \alpha_i} = -\frac{1}{2}\oint_{B_i}\frac{\partial y}{\partial g_k}$ which can be integrated w.r.t. g_k to end up with

$$\frac{\partial F}{\partial \alpha_i} = -\frac{1}{2}\oint_{B_i} y + H_i(\alpha_+, \alpha_j) \tag{3.18}$$

where $H_i(\alpha_+, \alpha_j)$ is an integration constant. The purpose of the next subsection is to fix this function *via* the superconformal anomaly.

3.4 Determination of the integration constant

In order to determine $H_i(\alpha_+, \alpha_j)$ we need to know the g_k-independent part of $F(g_k, \alpha_i, \alpha_+)$. This can be done with the help of the superconformal anomaly which can be computed in the microscopic U(N) theory using standard methods such as Pauli-Villars regularization. This leads to

$$D^{\dot{\alpha}} T_{\alpha\dot{\alpha}} = -(3N_c - N_a N_c) D_\alpha \hat{S}\,, \tag{3.19}$$

where $T_{\alpha\dot{\alpha}}$ is the $\mathcal{N} = 1$ supercurrent and $\hat{S}$ is the SU(N) glueball field. More precisely, we have

$$S_i = \hat{S}_i - \frac{1}{2N} w_{\alpha i} w^{\alpha i}\,. \tag{3.20}$$

This anomaly ought to be reproduced by the low energy effective potential $W_{\rm eff}(g_k, \hat{S}_i)$. The charges of $\hat{S}_i, g_k$ and θ_α are 3, $2-k$, and $-\frac{1}{2}$ respectively. We thus conclude that $W_{\rm eff}(g_k, \hat{S}_i)$ satisfies the equation

$$\left(\sum_k (2-k) g_k \frac{\partial}{\partial g_k} + 3\sum_i \hat{S}_i \frac{\partial}{\partial \hat{S}_i} - 3\right) W_{\rm eff} = -2N \sum_i \hat{S}_i \tag{3.21}$$

where we have assumed $N_a = 1$ and $N \equiv N_c$ in the last equality.[6] In order to see how this anomaly is reproduced in the low energy effective theory we need to relate the dynamical scale Λ_i of the low energy theory to the scale Λ of the U(N) theory [9]. We have

$$\begin{aligned} \Lambda_i^{3N_i} &= \Lambda^{2N} m_{\Phi_i}^{N_i} \prod_{j\neq i} m_{W_{ij}}^{-2N_j} \\ &= \Lambda^{2N} g_N^{N_i} \prod_{j\neq i} (z_i - z_j)^{N_i - 2N_j} \end{aligned} \tag{3.23}$$

where z_i are the roots of $W'(z)$. Taking into account the dimensions of g_k and z_i it is then not hard to see that the anomaly (3.21) is reproduced by

$$W_{\rm eff}(g_k, \hat{S}_i) = \sum_{i=1}^{N} \hat{S}_i \log\left(\frac{\Lambda_i^{3N_i}}{\hat{S}_i^{N_i}}\right) + P(g_k, \hat{S}_i) \tag{3.24}$$

where $P(g_k, \hat{S}_i)$ is a homogeneous polynomial in g_k and $\hat{S}_i$ transforming with weight 3. On the other hand, from (3.2) we have

$$W_{\rm eff}(g_k, \hat{S}_i) = \sum_i N_i \left.\frac{\partial F}{\partial S_i}\right|_{S_i = \hat{S}_i}. \tag{3.25}$$

[6] We note in passing that the superconformal anomaly equation can be combined with the Konishi anomaly $-\sum_k (k+1) g_k \frac{\partial}{\partial g_k} W = -2N\hat{S}$ to show that

$$\left(\sum_k g_k \frac{\partial}{\partial g_k} + \sum_i \hat{S}_i \frac{\partial}{\partial \hat{S}_i} - 1\right) W = 0 \tag{3.22}$$

i.e. $W_{\rm eff}(g_k, S_i)$ is a homogenous function of degree 1.

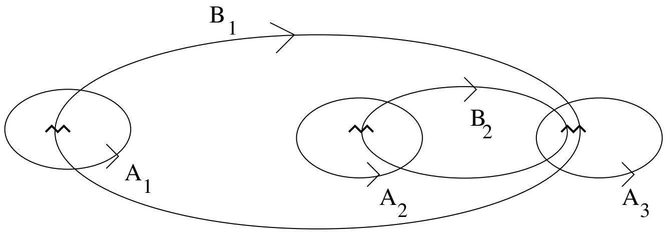

Figure 2: Riemann Surface Σ for U(3)

Let $F_0(S_i)$ be the g_k-independent part of $F(S_i, g_k)$. Then, eq. (3.24) and eq. (3.25) imply that $\frac{\partial F_0}{\partial S_i} = -S_i \log S_i + cS_i$ where c is an undetermined constant. Alternatively this gives for $1 \leq i \leq N-1$

$$\frac{\partial F_0}{\partial \alpha_i} = -S_i \log S_i + cS_i + S_N \log S_N - cS_N \tag{3.26}$$

where we have used the symmetry under the permutation of the S_i.

Let us then compare this result with the g_k-independent term in $-\frac{1}{2}\oint_{B_i} y$. This can be done using a scaling argument (see also [9]). Suppose we rescale all couplings g_k by λ, so that $W'(z) \to \lambda W'(z)$, $f(z) \to \lambda f(z)$ and consider $\tilde{S}_i = S_i/\lambda$ which is thus given by

$$\tilde{S}_i = -\frac{1}{2}\frac{1}{2\pi i}\oint_{A_i} \sqrt{W'^2 + \frac{f}{\lambda}}\,. \tag{3.27}$$

If we then let λ go to infinity, $\tilde{S}_i$ goes to zero. Geometrically this limit corresponds to vanishing A_i cycles since $\frac{f}{\lambda} \to 0$. Therefore we can rotate the two endpoints of the cut along one vanishing cycle A_i. Under such a transformation, S_i picks up a phase $e^{2\pi i}$. The transformation of the cycles can be worked out from figure 2: under a monodromy transformation $\tilde{S}_N \to e^{2\pi i}\tilde{S}_N$ all the B_i change to $B_i + A_N$ while under a monodromy transformation $\tilde{S}_i \to e^{2\pi i}\tilde{S}_i$, with $1 \leq i \leq N-1$, only B_i is changed in $B_i - A_i$. This implies the asymptotic form of the period integrals

$$-\frac{1}{2}\oint_{B_i} \sqrt{W'^2 + \frac{f}{\lambda}} \overset{\lambda\to\infty}{=} -\tilde{S}_i \log \tilde{S}_i + \tilde{S_N} \log \tilde{S}_N + \tilde{P}_i(\tilde{S}_j, \tilde{S}_N) \tag{3.28}$$

where $\tilde{P}_i(\tilde{S}_j, \tilde{S}_N)$ is an invariant polynomial of $\tilde{S_j}$ and $\tilde{S_N}$ with no constant term as the period integrals vanish in the limit $\lambda \to \infty$. Therefore, we get for large $\{g_k\}$ $(\lambda \to \infty)$

$$\begin{aligned} -\frac{1}{2}\oint_{B_i} y &= \lambda\left(-\frac{1}{2}\oint_{B_i}\sqrt{W'^2 + \frac{f}{\lambda}}\right) \\ &\underset{\lambda\to\infty}{=} -S_i \log\left(\frac{S_i}{\lambda}\right) + S_N \log\left(\frac{S_N}{\lambda}\right) + L_i(S_j, S_N) \end{aligned} \tag{3.29}$$

where $L_i(S_j, S_N)$ is a g_k-independent linear combination in S_j and S_N since only the linear terms of $\lambda\tilde{P}_i(\frac{S_j}{\lambda}, \frac{S_N}{\lambda})$ contribute in the limit $\lambda \to \infty$. On the other hand we see from (3.23) that rescaling of g_k by λ corresponds simply to rescaling of Λ_i so that we get

$$-\frac{1}{2}\oint_{B_i} y = -S_i \log\left(\frac{S_i}{\Lambda_i^3}\right) + S_N \log\left(\frac{S_N}{\Lambda_N^3}\right) + \tilde{c}(S_i - S_N) + \cdots \tag{3.30}$$

where the dots refer to terms which depend on g_k and where we have used the property that any choice of the basis of cycles A_i is physically equivalent.

Thus, the comparison of eq. (3.18) on one hand with eq. (3.26) and eq. (3.30) on the other hand give that the "integration constant" $H_i(\alpha_j, \alpha_+)$ is linear or more precisely that

$$\frac{\partial F}{\partial \alpha_i} = -\frac{1}{2}\oint_{B_i} y + d_1(S_i - S_N)\,. \tag{3.31}$$

The result (3.31) enables us then by use of eq. (3.14) to obtain the low energy couplings $\tau_{ij}(g_k)$ for the vector multiplets in terms of the period matrix of the Riemann surface Σ defined by anomalous chiral ring relations (2.17), that is

$$\tau_{ij}(g_k) = \left.\frac{\partial^2 F(g_k, \alpha_i, \alpha_+)}{\partial\alpha_i \partial\alpha_j}\right|_{\alpha_i = \bar{\alpha}_i} = 2\pi i \oint_{B_i} \xi_j + d_1\delta_{ij} + d_1\,, \tag{3.32}$$

where $\bar{\alpha}_i$ denotes the value on shell. In particular $\bar{\alpha}_+$ vanishes. Since the terms proportional to d_1 do not match the structure of the 1-loop correction, we have $d_1 = 0$. Furthermore, using the result $f(z) = d_0 g_N^2 \Lambda^{2N}$ on shell, Σ is given by

$$y^2(z) = W'^2(z) + d_0 g_N^2 \Lambda^{2N}\,. \tag{3.33}$$

The constant d_0 will be fixed in the next subsection.

3.5 $\mathcal{N} = 2$ limit and moduli of Σ

To complete our derivation of the low energy couplings for the massless U(1) vector multiplets w_α^i we first need to express the period matrix in eq. (3.32) in terms of the Casimir variables $u_k = \langle \operatorname{Tr}\phi^k \rangle$, $(1 \le k \le N)$, of the $\mathcal{N} = 2$ theory [4] and then fix the constant d_0 appearing in the equation for the Riemann surface Σ.

For this purpose, if ϕ_c is the classical value of ϕ obtained by extremizing the tree-level superpotential, we write $W'(z) = g_N P_N(z)$, where the coefficients of the polynomial $P_N(z)$ depend on $\operatorname{Tr}(\phi_c^k)$ through the relation

$$P_N(z) = \det(zId - \phi_c)\,. \tag{3.34}$$

We know however from section 2.3. that there are no quantum corrections to the Casimir variables i.e. that $u_k = \operatorname{Tr}(\phi_c^k)$. Thus, $P_N(z) = \langle \det(zId - \phi)\rangle$ and the equation (3.33) for Σ becomes $y^2 = g_N^2(P_N(z)^2 + d_0\Lambda^{2N})$ or equivalently

$$y^2 = \langle \det(zId - \phi)\rangle^2 + d_0\Lambda^{2N}\,. \tag{3.35}$$

The low energy couplings $\tau_{ij}(u_k)$ are thus given by the period matrix of the Riemann surface Σ defined by eq. (3.35). Furthermore, since the expectation values of the massless scalars are given by

$$a_i = \frac{1}{2\pi i}\oint_{A_i} \left\langle \operatorname{Tr}\left(\frac{z}{z - \phi}\right)\right\rangle, \tag{3.36}$$

JHEP08(2005)006

we get from eq. (2.6) together with eq. (3.35)

$$a_i = \frac{1}{2\pi i}\oint_{A_i}\frac{zP_N'(z)}{y(z)}\,. \tag{3.37}$$

The two equations (3.32) and (3.37) then determine the prepotential $\mathcal{F}(a_i,\Lambda)$ via $\tau_{ij}(a_i) = \frac{\partial^2}{\partial a_i \partial a_j}\mathcal{F}(a_i,\Lambda)$ up to physically irrelevant integration constants [34].

Finally let us fix the constant d_0. This can be done in several ways. Here we will fix d_0 by comparing the prediction for the observable $u_2(a_i)$ with the explicit one-instanton calculation. Indeed, we have on one hand from the superconformal Ward identity [27, 35, 5, 6]

$$\frac{iN}{\pi}u_2 = 2\mathcal{F} - \sum_{i=1}^{N-1} a_i\partial_{a_i}\mathcal{F}\,. \tag{3.38}$$

Upon substitution of the asymptotic instanton expansion for $\mathcal{F}(a_i)$ this then leads to [27, 36]

$$u_2 = \sum_i \phi_i^2 + 2\sum_{k=1}^{\infty} k\mathcal{F}_k(a_i)\left(-\frac{d_0}{4}\right)^k \Lambda^{2Nk}\,, \tag{3.39}$$

with [37] $\mathcal{F}_1(a_i) = \sum_{i=1}^N \prod_{j\neq i}\frac{1}{(\phi_i-\phi_j)^2}$. Here ϕ_i stands for the diagonal entries of $\langle\phi\rangle$. Eqn (3.39) is then a direct consequence of (3.32) and (3.33). On the other hand, comparing (3.39) with the explicit one-instanton calculation [3, 25, 38, 36] implies then $d_0 = -4$. Thus we have shown that the low energy couplings are indeed given by the period matrix of Σ with $d_0 = -4$.

4. Conclusions

In this article we have completed the field theory proof of the claim that the holomorphic effective action of $\mathcal{N} = 2$ SU(N) Yang-Mills action can be obtained by integrating a suitable anomaly. In particular, we have field the gap that arises in field theory (not present in string theory) due to the absence of special geometry relation between the periods of the Riemann surface Σ.

In addition we have pointed out that the anomalous chiral ring relations have important consequences for the multi-instanton calculus. Indeed, the instanton contributions to the expectation values of chiral ring elements are completely fixed by the 1-instanton observable $\langle\,\mathrm{Tr}(\Phi^{N-1}W^2)\rangle$. This also applies in a similar way to the expectation value $a_i = \frac{1}{2\pi i}\oint_{A_i}\langle\,\mathrm{Tr}(\frac{z}{z-\phi})\rangle$. On the other hand, expanding the contour integral in Λ one reads off the contribution of the n-instanton contribution to the function $a_i(g_k)$ (or equivalently $a_i(u_k)$). This seems to suggest a recursive structure that would be interesting to investigate within the multi-instanton calculus.

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JHEP08(2005)006

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A. Homogeneity of y

In this appendix, we prove that $y(z, g_k, S_i) = \sqrt{W'(z)^2 + f(z)}$ is homogeneous of degree 1 in S_i and g_k. For that purpose, we compute the various derivatives of y and compare the results obtained with the expression (3.13) of y found in section 3.2,

$$y = \sum_{k=0}^{N} g_k d\Omega_k - 4\pi i \sum_{i=1}^{N-1} \alpha_i \xi_i - \alpha_+ (2N\, d\Omega_+ + 4\pi i \sum_{i=1}^{N-1} \xi_i)\,. \tag{A.1}$$

The general strategy in the computations that will be done is to identify the holomorphic contributions. Let us recall at this stage that the space of holomorphic forms on the Riemann surface Σ is $N-1$ dimensional and two special basis are $\{\xi_i\}$ and $\{\frac{z^j}{2y}\}$ with $i = 1, \cdots, N-1$ and $j = 0, \cdots, N-2$. We change then the variables from (g_k, S_i) to (g_k, f_i) where $f_i(S_j, g_k)$ is the coefficient of order i of $f(z)$. Indeed, $\frac{\partial y}{\partial f_i} = \frac{z^i}{2y}$ is holomorphic except for $i = N-1$. However, it is easy to prove that

$$f_{N-1}(g_k, S_i) = -4 g_N \sum_{i=1}^{N} S_i \tag{A.2}$$

by computing the residue of y at infinity as it follows from the definition of y and the result (A.1) that it is respectively equal to $\frac{f_{N-1}}{2g_N}$ and to $-2\alpha_+$.

We first compute the derivative w.r.t. S_i. We have successively:

$$\begin{aligned} -\frac{1}{2}\left(\frac{\partial y}{\partial S_i}\right)_{S_j, g_k}(g_k, S_j) &= -\frac{1}{2}\sum_{j=0}^{N-1}\left(\frac{\partial y}{\partial f_j}\right)\left(\frac{\partial f_j}{\partial S_i}\right) \\ &= -\frac{1}{2}\sum_{j=0}^{N-2}\left(\frac{\partial y}{\partial f_j}\right)\left(\frac{\partial f_j}{\partial S_i}\right) - \frac{1}{2}\left(\frac{\partial y}{\partial f_{N-1}}\right)\left(\frac{\partial f_{N-1}}{\partial S_i}\right) \\ &= \sum_{j=0}^{N-2}\left(\frac{-z^j}{4y}\frac{\partial f_j}{\partial S_i}\right) + \frac{g_N z^{N-1}}{y}\,. \end{aligned} \tag{A.3}$$

The first contribution in the r.h.s. of (A.3) is a linear combination of holomorphic terms while the second term is a meromorphic form that has residue 1 at infinity.

Consider then the case $i = N$. In that case, we also have from the definition (2.18) of S_i

$$\oint_{A_i} -\frac{1}{2}\left(\frac{\partial y}{\partial S_N}\right) = 2\pi i \left(\frac{\partial S_i}{\partial S_N}\right)_{S_i, g_k} = 0$$

for $i = 1, \ldots, N-1$. Thus $-\frac{1}{2}\left(\frac{\partial y}{\partial S_N}\right)$ is a 1-form whose A_i integrals for $i = 1, \ldots, N-1$ vanish and which has residue 1 at infinity. However, there is an unique form satisfying

JHEP08(2005)006

these properties and by definition it is $d\Omega_+$. Thus, we have shown that

$$-\frac{1}{2}\frac{\partial y}{\partial S_N} = d\Omega_+ \,. \tag{A.4}$$

Take then $1 \le i \le N-1$. For $1 \le j \le N-1$, we get $\delta_{ij} = \frac{1}{2\pi i}\oint_{A_j} -\frac{1}{2}(\frac{\partial y}{\partial S_i})$. Thus, if we define the 1-form $k_i = -\frac{1}{2}\frac{\partial y}{\partial S_i} - 2\pi i \xi_i$ we have on one hand $\oint_{A_j} k_i = 0$ for $1 \le j \le N-1$ and on the other hand $\mathrm{Res}_\infty k_i = \mathrm{Res}_\infty(-\frac{1}{2}\frac{\partial y}{\partial S_i}) = 1$. Thus, for the same reason as above we have $k_i = d\Omega_+$ and so

$$-\frac{1}{2}\frac{\partial y}{\partial S_i} = 2\pi i \xi_i + d\Omega_+ \qquad \text{for } 1 \le i \le N-1\,. \tag{A.5}$$

It is now easy to compute the derivatives w.r.t. α_i and α_+ to get

$$\frac{\partial y}{\partial \alpha_i} = \frac{\partial y}{\partial S_i} - \frac{\partial y}{\partial S_N} = -4\pi i \xi_i\,, \tag{A.6}$$

$$\frac{\partial y}{\partial \alpha_+} = \sum_{i=1}^{N} \frac{\partial y}{\partial S_i} = -4\pi i \sum_{i=1}^{N-1} \xi_i - 2N\, d\Omega_+\,. \tag{A.7}$$

Let us compute now the derivative w.r.t. g_k. We have:

$$\begin{aligned}\frac{\partial y}{\partial g_k} &= \frac{z^k W'}{y} + \sum_{j=0}^{N-1} \frac{z^j}{2y}\frac{\partial f_j}{\partial g_k} \\ &= z^k\left(1 - \frac{f}{2(W')^2} + \cdots\right) + \sum_{j=0}^{N-1} \frac{z^j}{2y}\frac{\partial f_j}{\partial g_k} \\ &= z^k\left(1 - \frac{f_{N-1}}{2g_N^2 z^{N+1}} + o(z^{-N-1})\right) + \sum_{j=0}^{N-2} \frac{z^j}{2y}\frac{\partial f_j}{\partial g_k} + \frac{z^{N-1}}{2y}\frac{\partial f_{N-1}}{\partial g_k}\,.\end{aligned} \tag{A.8}$$

Suppose now that $0 \le k \le N-1$. Then, in eq. (A.8), the first term is equal to $z^k +$ (holomorphic terms), the second term is holomorphic and the third term vanishes by use of eq. (A.2). Thus, we have $\frac{\partial y}{\partial g_k} = z^k +$ (holomorphic terms). If $k = N$, it is easy to see that we come to the same conclusion thanks again to eq. (A.2). Furthermore,

$$\oint_{A_i} \frac{\partial y}{\partial g_k} = -4\pi i \frac{\partial S_i}{\partial g_k} = 0 \tag{A.9}$$

and thus

$$\frac{\partial y}{\partial g_k} = d\Omega_k\,. \tag{A.10}$$

We conclude that y is homogeneous of degree one in $(\alpha_i, \alpha_+, g_k)$ (and thus in (S_i, g_k)) by comparing eq. (A.1) with the results (A.6), (A.7) and (A.10).

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